Liquidity, Asset Prices and Financial Instability*

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Abstract

This paper considers the stability of financial systems. We study a model where banks provide liquidity insurance and interact on asset markets. Coordination failures among depositors can cause simultaneous bank runs and asset market crashes. With their portfolio decisions, banks choose whether to be vulnerable to bank runs. Depositors choose where to make their deposits knowing the portfolio decisions of banks. In equilibrium both the share of banks that are vulnerable to bank runs and the volatility of asset prices are endogenous. There can be multiple equilibria and even indeterminacy. Fundamentals can be compatible with stable financial systems where no bank is vulnerable to bank runs and asset prices are stable as well as with weak financial systems where some banks or even all banks are vulnerable to bank runs and asset prices are volatile. We compare different financial systems with respect to their real economic implications.

Keywords Liquidity Insurance · Extrinsic Uncertainty · General Equilibrium · Bank Run · Asset Price Volatility

JEL Classification

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1 Introduction

From time to time a significant number of banks fail and asset markets crash simultaneously. Such financial crisis can be caused by changes in fundamentals hitting a financial system which reacts heavily. Another possibility is that a crisis originates in the financial system without changes in fundamentals. In that scenario, what are the real economic implications of the crisis?

In the present paper we study economies with no intrinsic uncertainty where banks make productive investments, provide liquidity insurance and interact on asset markets. There is extrinsic uncertainty in that depositors can fail to coordinate. Indeed depositors can engage in bank runs that can lead to bank failures and asset market crashes. For banks to be immune to bank runs they need to follow strategies that enable them to satisfy demands of depositors regardless whether depositors run or not. Financial systems are described along two dimensions, namely the share of banks prone to bank runs and the volatility of asset prices. For the two extremes, financial systems are *stable* provided no bank is prone to bank runs and asset prices are constant and *unstable* provided all banks are prone to bank runs and asset prices are volatile. Financial systems are *weak* provided they are not stable. An economy can have multiple financial systems or there is even indeterminacy. For economies with multiple financial systems, one can be stable, one unstable and others weak. For economies with indeterminacy, the financial systems can consist only of banks which are immune to bank runs while asset price volatility is indeterminate.

Our starting point is a variation of the Diamond and Dybvig (1983) model as in Allen and Gale (2004a,b). There are three dates and a continuum of ex-ante identical consumers, who live for two or three dates. Every consumer has one unit of a good and is able to store it between dates. Banks can make productive investments and hold reserves by storing goods. Productive investments involve an investment at the first date and generate an output at the final date. Banks provide liquidity insurance by offering simple deposit contracts according to which consumers can withdraw their deposits whenever they want, but the rate of return can depend on the date of withdrawal. Banks can trade productive investments at the second date.

We differ from Allen and Gale (2004a,b) in that deposit contracts make banks susceptible to bank runs even though there is no intrinsic uncertainty. Suppose banks cannot satisfy demands of all depositors in case they run at the second date. Then coordination failures are possible: if all depositors believe all other depositors will run, then they will run themselves. For banks to be immune to bank runs they need to ensure their portfolio is always liquid and they do so by extending their reserve ratio. These banks are safe. Banks prone to bank runs are risky banks. For them the upside is that they do not need to make sure their portfolio is as liquid. The downside is that the return to depositors is limited in case they run. We consider two extrinsic states. In the first state all depositors run provided their banks cannot satisfy the demands of all depositors in full at the second date. In the second state nobody runs.

In equilibrium banks offer deposit contracts and consumers choose where to make deposits. The banking sector is perfectly competitive. Therefore every bank offers a contract that maximizes expected utility of consumers and every consumer makes a deposit in the bank she prefers. Should banks offering different contracts coexist, then consumers are to be indifferent between these contracts even though some can be immune to bank runs and others are run-prone. Since portfolio decisions are made before the extrinsic states are know, aggregate availability of reserves in those states is inelastic, implying cash-in-the-market pricing and asset prices being potentially different across states. We carry out a detailed study of equilibria. They are shown to exist (Theorem 1). While the proof does not tell anything about their properties, there can be only three types of equilibria: solely safe banks operate so asset prices ensure that banks do not want to buy assets; solely risky banks operate so asset prices have to ensure that risky banks do not want to sell assets; and, there are both risky and safe banks so asset prices ensure asset supply of risky banks is equal to asset demand of safe banks.

We show that if the probability of coordination failures is sufficiently small, there may be only risky banks and the asset price drops to or below the physical liquidation value of assets when bank runs occur (Theorem 2). If the probability of coordination failures is sufficiently large, there may be only safe banks and equilibrium asset prices are deterministic (Theorem 3). If a stable financial system exists where safe banks strongly dominate risky banks, there is indeterminacy because such equilibrium is not locally isolated (Theorem 4).

Some simple examples show that there can also be multiple, locally-isolated equilibria. For instance, economies can have weak financial systems and stable financial systems, or unstable financial systems can exist along with stable financial systems. We take such economies with multiple equilibria as a starting point to study the real economic implications of financial instability. The advantage of this approach is that we compare economies with the same fundamentals and the same probability of coordination failures and the only difference is the endogenous instability of their financial systems. Therefore, any differences in the real outcomes can be attributed solely to the difference in financial instability. Our approach does thus not simply look how financial systems dampen or amplify shocks to fundamentals, or how strong the backlash in the real economy is once a financial crisis occurs, or how the allocation of funds is distorted in anticipation of crises — for those differences could be in principle attributed to differences in the primitives that give rise to different financial systems in the first place. The model suggests that equilibria differ according to the value of the liquidity insurance consumers obtain. They also differ with respect to how funds are allocated across liquid reserves and illiquid productive capital investments. For non-increasing relative risk aversion we find that among the financial systems that can emerge, the value of liquidity insurance and the share of savings going into productive investments are both the larger the higher is the asset price in the crisis state (Theorem 5). We provide an example of an overlapping generations economy with production which can grow stronger with a stable financial system than with a banking sector consisting of both risky and safe banks.

The idea of sunspots affecting real outcomes goes back to Cass and Shell (1983). That bank runs can be triggered by sunspots has been first suggested by Diamond and Dybvig (1983) and then scrutinized in-depth from a mechanism design perspective (e.g. Jacklin, 1987; Green and Lin, 2003). We restrict attention to simple deposit contracts, but according to Peck and Shell (2003) and Sultanum (2014) even an optimal contract implies the possibility of a bank run driven by coordination failures.

There is a large literature on the interactions of asset markets and liquidity-providing banks. There, crises are often treated as zero probability events (e.g. Fecht, 2004) or the level of liquidity held by banks is exogenous (e.g. Brunnermeier and Pedersen, 2005; Diamond and Rajan, 2005). Allen and Gale (2004a,b) analyze economies with positive crisis probabilities and endogenous liquidity holdings by banks. There, bank failures only occur when shocks to fundamentals drive the market value of a bank's assets below the present value of its promised payments to depositors. There are no coordination failures. Bank runs are thus primarily solvency rather than liquidity problems, which is our focus. An important insight from Allen and Gale (2004a) is that fundamental shocks can have disproportionately large effects on banks and asset prices. However, in the limit economy where fundamentals become asymptotically deterministic, the equilibrium converges to one in which banks will never default and provide an unconstrained efficient liquidity insurance. Accordingly, while the financial system has a tendency to react strongly to real shocks, infinitesimal shocks have no real economic implications. We consider the case of asymptotically deterministic fundamentals as the starting point for our analysis. For example, one could think of our extrinsic risk as some fundamental with values that differ only to some infinitesimal extent in two states. The possibility of coordination failures imposes a tighter constraint on banks to be safe than in Allen and Gale (2004a). This prevents safe banks from providing optimal liquidity insurance even in absence of fundamental risks. Therefore, any financial system with at least some banks being safe deviates from optimum liquidity insurance. Risky banks choose not to be subject to such additional constraint. Accordingly, only unstable financial systems offer liquidity insurance that converges to the first-best provided the probability of sunspots approaches zero. Moreover, despite there being no fundamental risk, there can be multiple equilibria and even indeterminacy in real terms.

Bencivenga and Smith (1991), Ennis and Keister (2003) and Fecht et al. (2008) analyze the role of banks providing liquidity insurance for capital formation and growth. In Bencivenga and Smith (1991) there are no bank runs and no asset markets. In Ennis and Keister (2003) and Fecht et al. (2008) banks as well as some consumers can trade on asset markets but equilibria are symmetric. Ennis and Keister (2003) allow for coordination failures, but bank runs inevitably lead to physical liquidation of capital since the set of equilibria is restricted to symmetric equilibria, rendering asset prices deterministic. In our paper, asset markets can prevent inefficient liquidation. In all but unstable financial systems, productive assets originated by banks will change hands in the course of a banking crisis. This is because of two features of such financial systems. First, there are some safe banks, which by their very nature hold excess reserves. Second, asset prices clear the market in all states.

Therefore, while the possibility of a crisis and its expected intensity affect the ex-ante allocation of resources and thus impose an externality on future generations of consumers, the actual occurrence of a crisis does not. Only with an unstable financial system there will be no buyers of assets in the market and the actual occurrence of a financial crisis itself imposes an externality on future generations. In Fecht et al. (2008) there are neither extrinsic nor intrinsic risks. Their focus is on the role of the direct participation of consumers in asset markets for the trade-off between capital formation and insurance of idiosyncratic liquidity risks. In our model, such trade-off crucially depends on which asset prices prevail in equilibrium.

In Acharya et al. (2011), Brunnermeier and Sannikov (2014) and Diamond and Rajan (2011) safe banks do not only hold liquidity to prevent a bank run, but also to strategically take advantage of fire sales of distressed competitors. In our model, relatively strong risk aversion of consumers looking for liquidity insurance together with perfect competition among banks prevent banks from such speculation on fire sales in equilibrium. Similar to Lin et al. (2016) default is a necessary condition for determinacy of the real outcome in our model. While the focus there is on the role of default in an economy with money, ours is about the role of default in an economy with banks.

The paper has the following structure. In section 2 we lay out the model. In section 3 we show that equilibria exist and describe the key dimensions along which equilibria can be differentiated. In section 4 we look into the role of the sunspot probability and other key parameters for the type of equilibria that can emerge. In section 5 we discuss some consequences of the instability of the financial system for the real economy. Section 6 concludes.

2 The model

2.1 Setup

There are three dates $t \in \{0,1,2\}$, and at each date there is a single good. There is a continuum of identical consumers with mass one. A consumer is described by her endowment $(1,0,0)$, consumption set $X = \mathbb{R}^2_+$ and random utility function $U : X \mapsto \mathbb{R}$. A consumer is either impatient and values consumption only at date $t = 1$, or patient and values consumption only at date $t = 2$. At date $t = 1$ consumers learn their type, which is private information. Patience among consumers is uncorrelated and the share of impatient consumers is λ. Therefore,

$$U(x_1,x_2) = \begin{cases} u(x_1) & \text{with probability} \quad \lambda \in]0,1[, \\ u(x_2) & \text{with probability} \quad 1-\lambda. \end{cases} \tag{1}$$

The elementary utility function u is twice differentiable with $u' > 0$, $u'' < 0$, and $\lim_{x \to 0} u'(x) = \infty$. The relative risk aversion is supposed to satisfy $k(x) = -\frac{u''(x)}{u'(x)}x > 1$.

There are two constant-returns-to-scale technologies, or assets. A short asset (reserves) has a gross return of 1 between t and $t+1$ for $t \in \{0,1\}$. A long asset (investment) is originated at date $t=0$ and generates $R>1$ at $t=2$. Before maturity it can be unwound for some positive, but arbitrarily small ε (physical liquidation value). Consumers cannot make investments, only banks can.

There is a continuum of banks. Each bank supplies financial services to a representative fraction of consumers. It does so by offering a deposit contract to consumers in exchange for their endowment. It then puts a share $y \in [0,1]$ of these endowments in the short asset as reserves and invests $1-y$ in the long asset. At $t=1$ banks can trade among each other the investments for reserves at a price P on an asset market. The deposit contract allows consumers to withdraw d at $t=1$. Those who do not withdraw will equally share the residual value of their bank's assets at $t=2$. Each consumer is free to choose a bank at $t=0$, but cannot do business with more than one bank or change the bank after that date. Perfect competition among banks ensures that they maximize consumers' ex ante expected utility.

There is an extrinsic risk. We assume that the economy is in one of two states $s \in \mathbb{S} = \{1,2\}$ at date $t=1$. With probability $p \in]0,1[$, the state is $s=1$ in which patient consumers contemplate to run on their bank: a patient consumer will compare what she gets by withdrawing at $t=1$ with the payoff associated with holding on until $t=2$, assuming that all other patient consumers will withdraw at $t=1$. If the former is higher, it is best for everyone to withdraw at $t=1$. A bank run occurs as a result of self-fulfilling expectations. The bank has to sell or liquidate all assets, and the proceeds are equally shared among all its consumers at $t=1$. If state $s=2$ materializes, a patient consumer assumes that all other patient consumers will not withdraw at date $t=1$, hence there is no such coordination failure and a bank run does not happen. It is not possible to write state-contingent contracts and financial markets are incomplete.

2.2 Bank behavior

At date $t=0$ banks can either take their chances, or they make provisions to prevent a possible bank run. Accordingly, banks are either risky or safe. In state $s=1$, the run on risky banks forces them to cease operating at $t=1$, while in state $s=2$ risky and safe banks live on until date $t=2$.

Let P_s be the price of the long asset at date $t=1$ in state s and $x=(x_{1,1},x_{1,2},x_{2,1},x_{2,2})$ denote the bundle of consumption x_{ts} at date t in state s. A bank's objective then is to maximize expected utility

$$\max_{(y,d,x)} \quad \lambda\left(pu\left(x_{1,1}\right)+(1-p)u\left(x_{1,2}\right)\right)+(1-\lambda)\left(pu\left(x_{2,1}\right)+(1-p)\,u\left(x_{2,2}\right)\right) \tag{2}$$

subject to its constraints. These constraints are different for safe and risky banks. For a bank to be safe, the market value of its assets must at least cover the total value of outstanding deposits as of $t=1$, i.e.

$$d \leq y+P_s\,(1-y) \quad \forall s \in \mathbb{S}. \tag{3}$$

This implies that the bank holds more liquidity than needed to payout impatient consumers. However, since $k(x) > 1$, a safe bank does not hold more than needed to deter consumers from running. Consumers are simply too risk averse to be interested in speculating on fire-sales, as this would only benefit patient consumers at the expense of impatient consumers.[1] Hence

$$\begin{aligned} d &= y + P_s(1-y) \quad \text{if} \quad 1 - P_s > 0, \\ d &< y + P_s(1-y) \quad \text{if} \quad 1 - P_s < 0. \end{aligned} \tag{4}$$

The resource constraints on consumption with a safe bank are

$$x_{1,s} \leq d, \tag{5a}$$

$$\lambda d + (1-\lambda)\frac{x_{2,s}}{R/P_s} \leq y + P_s(1-y). \tag{5b}$$

The first constraint reflects that a safe bank can always repay its deposits at date $t = 1$. The second requires that the present value of total consumption equals the present value of the bank's assets.

Let superscript $\mathscr{S}$ denote the solutions to a safe bank's problem. As the problem is convex, it is unique and, if interior, solves the first-order condition

$$\left(\frac{1}{R}\frac{\lambda}{1-\lambda}u'\left(x_{1,1}^{\mathscr{S}}\right) + \frac{p}{P_1}u'\left(x_{2,1}^{\mathscr{S}}\right)\right)(1-P_1) - \frac{1-p}{P_2}u'\left(x_{2,2}^{\mathscr{S}}\right)\left((P_2-1) + \frac{\lambda}{1-\lambda}(P_2-P_1)\right) = 0, \tag{6}$$

implying that an incentive constraint $x_{1,s}^{\mathscr{S}} \leq x_{2,s}^{\mathscr{S}}$ is never binding.

As for a risky bank, there is a run in state $s = 1$ if the market value of the bank's assets is not sufficient to fully pay all depositors

$$y + P_1(1-y) \leq d. \tag{7}$$

Two remarks are due. First, in equilibrium condition (7) will always hold with strict inequality. This is because otherwise a safe bank strictly dominates a risky bank. Second, a risky bank cannot fail in both states. Otherwise the marginal rate of substitution between early and late consumption would be 1, regardless in which state s the economy is, while the ex-ante marginal rate of transformation is R^{-1}. This cannot be optimal either.

[1] See Appendix A.

The resource constraints on consumption with a risky bank read

$$x_{1,s} \leq \begin{cases} y + P_1(1-y) & \text{if} \quad s=1, \\ d & \text{if} \quad s=2, \end{cases} \tag{8a}$$

$$x_{2,s} \leq \begin{cases} y + P_1(1-y) & \text{if} \quad s=1, \\ \dfrac{R}{P_2}\dfrac{y + P_2(1-y) - \lambda d}{1-\lambda} & \text{if} \quad s=2. \end{cases} \tag{8b}$$

The first lines in these budget constraints reflect that in a bank run everyone gets a pro-rata share of the bank's liquidation value. The second lines state that impatient consumers get what the deposit contract entitles them to, while patient consumers share the remainder of the bank's revenue at $t = 2$.

Let superscript $\mathscr{R}$ denote the solution to a risky bank's problem. As the problem is convex, the solution is unique. If it is interior, it solves the first-order conditions

$$\frac{u'\left(x_{1,2}^{\mathscr{R}}\right)}{u'\left(x_{2,2}^{\mathscr{R}}\right)} - \frac{R}{P_2} = 0, \tag{9a}$$

$$\frac{u'\left(x_{1,1}^{\mathscr{R}}\right)}{u'\left(x_{2,2}^{\mathscr{R}}\right)} - \frac{1-p}{p}\frac{P_2 - 1}{1-P_1}\frac{R}{P_2} = 0, \tag{9b}$$

implying that an incentive constraint $x_{1,s}^{\mathscr{R}} \leq x_{2,s}^{\mathscr{R}}$ is never binding.

At date $t = 1$, banks trade investments for reserves. Risky banks either liquidate or sell everything in the sunspot state, otherwise they sell or buy provided the payments to their impatient consumers can be made. A safe bank holds reserves in (weak) excess of what it actually owes to depositors, regardless in which state the economy is. Liquidity demand q^D of a single risky bank (supply of investments) and liquidity supply q^S of a single safe bank (demand for investments) are

$$q_s^D = \begin{cases} P_1\left(1 - y^{\mathscr{R}}\right) & \text{if} \quad s=1, \\ \left(\lambda d^{\mathscr{R}} - y^{\mathscr{R}}\right) & \text{if} \quad s=2, \end{cases} \tag{10a}$$

$$q^S = y^{\mathscr{S}} - \lambda\left(y^{\mathscr{S}} + P_1\left(1 - y^{\mathscr{S}}\right)\right). \tag{10b}$$

Let ρ be the share of consumers who have put their endowments in risky banks, or the share of risky banks for short. Then,

$$Q_s^D = \rho q_s^D, \tag{11a}$$

$$Q^S = (1-\rho)q^S, \tag{11b}$$

denote aggregate liquidity demand and aggregate liquidity supply, respectively. The possibility of coordination failures among consumers implies that, for a bank to be safe, it has to hold more short assets relative to what it can promise to impatient consumers. Everything else equal, this puts safe banks at a greater disadvantage than in Allen and Gale (2004a).

3 Equilibrium

It is convenient to further simplify notation. First, a consumption plan for a consumer who deposits her savings with a bank of type $\tau \in \{\mathscr{S}, \mathscr{R}\}$ is a consumption bundle and a bank portfolio (x^τ, d^τ, y^τ) satisfying the constraints (3), (5a) and (5b), or (7), (8a) and (8b), respectively. Second, let $P = (P_1, P_2)$. Then, $V^\tau(P)$ denotes the indirect utility offered to consumers by a bank of type τ. Finally, the full-information first-best is characterized by reserve holdings y^* defined by

$$\frac{u'(y^*/\lambda)}{u'\left(\frac{R(1-y^*)}{1-\lambda}\right)} = R. \tag{12}$$

We can now define our equilibrium concept.

Definition 1 *An* ***equilibrium*** *is a set of consumption plans, asset prices and the share of risky banks*

$$\left((y^{\mathscr{S}}, d^{\mathscr{S}}, x^{\mathscr{S}}), (y^{\mathscr{R}}, d^{\mathscr{R}}, x^{\mathscr{R}}), (P, \rho)\right)$$

with the following properties:

- *Banks maximize expected utility:* $(y^{\mathscr{S}}, d^{\mathscr{S}}, x^{\mathscr{S}})$ *is a solution to the consumer problem for safe banks, and* $(y^{\mathscr{R}}, d^{\mathscr{R}}, x^{\mathscr{R}})$ *is a solution to the consumer problem for risky banks.*

- *Markets clear:*

$$\begin{aligned} \rho P_1\left(1-y^{\mathscr{R}}\right) + (1-\rho)\left(\lambda\left(y^{\mathscr{S}} + P_1\left(1-y^{\mathscr{S}}\right)\right) - y^{\mathscr{S}}\right) &= 0 \quad \textit{if} \quad s = 1, \\ \rho\left(\lambda d^{\mathscr{R}} - y^{\mathscr{R}}\right) + (1-\rho)\left(\lambda\left(y^{\mathscr{S}} + P_1\left(1-y^{\mathscr{S}}\right)\right) - y^{\mathscr{S}}\right) &= 0 \quad \textit{if} \quad s = 2. \end{aligned}$$

- *Consumers are not better off by going to another operating bank:*

$$\begin{aligned} V^{\mathscr{S}}(P) &= V^{\mathscr{R}}(P) \quad \textit{if} \quad \rho \in]0,1[, \\ V^{\mathscr{S}}(P) &\geq V^{\mathscr{R}}(P) \quad \textit{if} \quad \rho = 0, \\ V^{\mathscr{S}}(P) &\leq V^{\mathscr{R}}(P) \quad \textit{if} \quad \rho = 1. \end{aligned}$$

3.1 Existence

We begin with the fundamental question of the existence of equilibria.

Theorem 1 *There is an equilibrium.*

Proof: For the functions $M,N:\mathbb{R}_{++}\to\mathbb{R}_{++}$ defined by $M(P)=\max\{R/P,1\}$ and $N(P)=\max\{P,\varepsilon\}$ consider the consumer problem

$$\max_{(y,d,x)} \quad \lambda(pu(x_{1,1})+(1-p)u(x_{1,2}))+(1-\lambda)(pu(x_{2,1})+(1-p)u(x_{2,2}))$$

$$\text{s.t.}\quad \begin{cases} \left.\begin{array}{lcl} x_{1,1} & \leq & d \\ x_{1,2} & \leq & d, \\ x_{2,1} & \leq & M(P_1)(y+N(P_1)(1-y)-\lambda d) \\ x_{2,2} & \leq & M(P_2)(y+N(P_2)(1-y)-\lambda d) \end{array}\right\} \text{for } \begin{array}{lcl} y+N(P_1)(1-y) & \geq & d \\ y+N(P_2)(1-y) & \geq & \lambda d, \end{array} \\ \left.\begin{array}{lcl} x_{1,1} & \leq & y+N(P_1)(1-y) \\ x_{1,2} & \leq & d, \\ x_{2,1} & \leq & y+N(P_1)(1-y) \\ x_{2,2} & \leq & M(P_2)(y+N(P_2)(1-y)-\lambda d) \end{array}\right\} \text{for } \begin{array}{lcl} y+N(P_1)(1-y) & < & d \\ y+N(P_2)(1-y) & \geq & \lambda d. \end{array} \\ \quad y \in [0,1] \end{cases}$$

For all $(P_1,P_2)\in\mathbb{R}^2_{++}$ there is a solution because the set of alternatives is compact. According to Berge's maximum theorem the solution correspondence $F:\mathbb{R}^2_{++}\to[0,1]\times\mathbb{R}_+\times\mathbb{R}^4_+$ is upper hemi-continuous because expected utility is a continuous function and the set of alternatives is a continuous correspondence.

Let the correspondence $G:\mathbb{R}^2_{++}\to\mathbb{R}^2$ be defined by: in case $(y,d,x)\in F(P_1,P_2)$ satisfies the first four budget constraints,

$$G_s(P_1,P_2) = \begin{cases} \dfrac{y+\varepsilon(1-y)-\lambda d}{P_s} & \text{for } P_s<\varepsilon, \\ \left[\dfrac{y+P_s(1-y)-\lambda d}{P_s},\dfrac{y-\lambda d}{P_s}\right] & \text{for } P_s=\varepsilon, \\ \dfrac{y-\lambda d}{P_s} & \text{for } \varepsilon<P_s<R, \\ \left[\dfrac{y-\lambda d}{P_s},-(1-y)\right] & \text{for } P_s=R, \\ -(1-y) & \text{for } P_s>R, \end{cases}$$

for both s; or, in case (y,d,x) satisfies the second four budget constraints,

$$G_1(P_1,P_2) \;=\; \begin{cases} 0 & \text{for } P_1 < \varepsilon, \\ [-(1-y),1-y] & \text{for } P_1 = \varepsilon, \\ -(1-y) & \text{for } P_1 > \varepsilon, \end{cases}$$

and $G_2(P_1,P_2)$ as in case (y,d,x) satisfies the first four budget constraints. Then G is upper hemi-continuous.

For $(P_1,P_2) \in \mathbb{R}^2_{++}$ and $(y,d,x) \in F(P_1,P_2)$, if $P_s < \varepsilon$ and $(z_1,z_2) \in G(P_1,P_2)$, then $z_s \geq 0$. For $(P_1,P_2) \in \mathbb{R}^2_{++}$ and $(y,d,x) \in F(P_1,P_2)$, if $P_s > R$ and $(z_1,z_2) \in G(P_1,P_2)$, then $z_s \leq 0$. Therefore prices are bounded from below by $\varepsilon-\delta$ and from above by $R+\delta$ for some $\delta \in]0,\varepsilon[$, $(P_1,P_2) \in [\varepsilon-\delta,R+\delta]^2$.

For $A \subset \mathbb{R}^2$ being the convex hull of the range of G with prices restricted to the set $[\varepsilon-\delta,\mathbb{R}+\delta]^2$,

$$A \;=\; \operatorname{co}\{\,(z_1,z_2) \in \mathbb{R}^2 \mid \exists (P_1,P_2) \in [\varepsilon-\delta,R+\delta]^2 : (z_1,z_2) \in G(P_1,P_2)\,\}$$

let the correspondence $H : A \to [\varepsilon-\delta,R+\delta]^2$ be defined by

$$H(z_1,z_2) \;=\; \{\,(P_1,P_2) \in [\varepsilon-\delta,R+\delta] \mid \forall (P_1',P_2') \in [\varepsilon-\delta,R+\delta] : P_1z_1+P_2z_2 \geq P_1'z_1+P_2'z_2\,\}.$$

Then H is upper hemi-continuous.

The correspondence $(\operatorname{co} G,H) : [\delta,R+\delta]^2 \times A \to [\delta,R+\delta]^2 \times A$ has a fixed point according to Kakutani's fixed point theorem, because $[\varepsilon-\delta,R+\delta]^2 \times A$ is convex and compact and $(\operatorname{co} G,H)$ is convex valued and upper hemi-continuous. Suppose $(P_1,P_2,z_1,z_2) \in [\varepsilon-\delta,R+\delta]^2 \times A$ is a fixed point, so $(z_1,z_2) \in \operatorname{co} G(P_1,P_2)$ and $(P_1,P_2) \in H(z_1,z_2)$. Suppose $z_s \neq 0$, then $H_s(z_1,z_2) = \varepsilon-\delta$ in case $z_s < 0$ and $H_s(z_1,z_2) = R+\delta$ in case $z_s > 0$. Suppose $P_s = \varepsilon-\delta$, then either $z_s = 0$ or $z_s > 0$ contradicting $P_s = \varepsilon-\delta$, so $z_s = 0$. If $P_s = R+\delta$, then either $z_s = 0$ or $z_s < 0$ contradicting $P_1 = R+\delta$, so $z_s = 0$. Therefore $z_s = 0$ for both s.

For every $(z_1,z_2) \in \operatorname{co} G(P_1,P_2)$ there are at most three points $(z_1^i,z_2^i)_i$ with $(z_1^i,z_2^i) \in G(P_1,P_2)$ for every i and at most three weights $(w^i)_i$ with $w^i > 0$ for every i and $\sum_i w^i = 1$ such that $(z_1,z_2) = \sum_i w^i(z_1^i,z_2^i)$ according to Caratheodory's theorem. Hence (P_1,P_2,z_1,z_2) is an equilibrium. □

An equilibrium always exist, although solving for an equilibrium is difficult. However, we can say something about equilibrium asset prices and about the structure of the banking sector. As regards asset prices, one characteristic is that there cannot be arbitrage opportunities. At date $t=0$ banks have access to two assets with identical costs: the long asset with values (P_1,P_2) and the short asset with values $(1,1)$, both at date $t=1$. The assets are arbitrage free if and only if $P_1 < 1 < P_2$, $P_2 < 1 < P_1$ or $P_1 = P_2 = 1$. If $P_1,P_2 \geq 1$ with $P_1+P_2 > 2$, then all banks would solely invest in the long asset.

However consumers are better off with a mix of long and short assets. If $P_1, P_2 \leq 1$ with $P_1 + P_2 < 2$, then all banks would solely invest in the short asset. Consumers can do so on their own without using banks, hence banks have a mix of long and short assets. Finally $P_1, P_2 \leq R$ because otherwise all banks would sell all their long assets in the state s with $P_s > R$ and nobody would be willing to buy the long asset. In addition to these no-arbitrage conditions, prices must satisfy $P_1 \leq P_2$. This is because risky banks sell all their long assets in state $s = 1$ and (weakly) fewer assets in state $s = 2$ whereas the supply of liquidity from safe banks is identical in both states.

As regards the structure of the banking sector, there are potentially three types of equilibria. There may exist only risky banks ($\rho = 1$) or only safe banks ($\rho = 0$) or a mix of safe and risky banks ($\rho \in]0,1[$). Accordingly, to characterize any equilibrium, three conditions need to be analyzed. The first condition refers to the state-independence of liquidity demand, which is necessary in any equilibrium with $\rho > 0$ because liquidity supply is state-independent and so has to be its demand in any equilibrium with risky banks. The second condition refers to zero liquidity supply, which is necessary in any equilibrium with $\rho = 0$, for there will be no liquidity demand. The third is about a consumer's choice between banks of different types.

3.2 State-independent liquidity demand

State-independence of liquidity demand, i.e. $q_1^D = q_2^D$, requires $d^{\mathscr{R}} = \left(y^{\mathscr{R}} + P_1(1-y^{\mathscr{R}})\right)/\lambda$. To derive the feasible pairs of prices P that induce risky banks to find it optimal to set $y^{\mathscr{R}}$ and $d^{\mathscr{R}}$ such that liquidity demand is state independent, we define a correspondence f such that for $P_1 \in [\varepsilon, 1]$

$$f(P_1) = \begin{cases} \left\{(y^{\mathscr{R}}, P_2) \in \{0\} \times [1,R] \;\middle|\; \left(y^{\mathscr{R}}, d^{\mathscr{R}}\right) \text{ satisfy (9a) and } d^{\mathscr{R}} = P_1/\lambda \right\}, \\ \left\{(y^{\mathscr{R}}, P_2) \in]0,1] \times [1,R] \;\middle|\; \left(y^{\mathscr{R}}, d^{\mathscr{R}}\right) \text{ satisfy (9a), (9b) and } d^{\mathscr{R}} = \left(y^{\mathscr{R}} + P_1(1-y^{\mathscr{R}})\right)/\lambda \right\}. \end{cases} \tag{13}$$

If $f(P_1) = \emptyset$, then P_1 is incompatible with state-independent liquidity demand. For $f(P_1) \neq \emptyset$, let $(\mathbf{y}^{\mathscr{R}}, \mathbf{P}_2)$ denote a solution to Equation (13) such that $(y^{\mathscr{R}}, d^{\mathscr{R}})$ is a solution to a risky bank's optimization problem and liquidity demand is state independent provided $y^{\mathscr{R}} = \mathbf{y}^{\mathscr{R}}$ and $d^{\mathscr{R}} = \left(P_1(1-\mathbf{y}^{\mathscr{R}}) + \mathbf{y}^{\mathscr{R}}\right)/\lambda$. In principle, there can be many solutions for a given P_1. For any of them, Equation (13) defines P_2 as an implicit function of P_1 in some neighborhood of $(\mathbf{y}^{\mathscr{R}}, \mathbf{P}_2)$ according to the general implicit function theorem. Let $k_{ts} = k(x_{t,s}^{\mathscr{R}})$ denote the relative risk aversion at $x_{t,s}^{\mathscr{R}}$. Then, for each solution this implicit function satisfies

$$\frac{dP_2}{dP_1} = \begin{cases} \dfrac{k_{22} + \left(\frac{P_2}{P_1} - 1\right) k_{11}}{k_{22} + \left(\frac{P_2}{P_1} - 1\right)} \dfrac{P_2}{P_1} & \text{for } y^{\mathscr{R}} = 0, \\ -\dfrac{(k_{11} - k_{12}) k_{22} \frac{P_2 - 1}{P_2 - P_1} + k_{12} + k_{22} \frac{y^{\mathscr{R}} + P_1(1 - y^{\mathscr{R}})}{(1 - P_1)(1 - y^{\mathscr{R}})}}{(k_{11} - k_{12}) k_{22} \frac{P_1}{P_2 - P_1} + k_{12} \frac{1}{P_2 - 1} + k_{22} \frac{y^{\mathscr{R}} + P_1(1 - y^{\mathscr{R}})}{(1 - P_1)(1 - y^{\mathscr{R}})} \frac{P_2}{P_2 - 1} + k_{11}} \dfrac{P_2}{1 - P_1} & \text{for } y^{\mathscr{R}} > 0. \end{cases} \tag{14}$$

Our first observation is about some characteristics of the relation between prices provided liquidity demand is state-independent.

Lemma 1 *Assume relative risk aversion is non-increasing. Provided $f(P_1) \neq \emptyset$, state-independent liquidity demand implies that the asset price in state $s = 2$, P_2, is well-defined by the asset price in state $s = 1$, P_1. For $y^{\mathscr{R}} > 0$, P_2 is monotonically increasing in P_1. For $y^{\mathscr{R}} = 0$, P_2 is monotonically decreasing in P_1.*

Proof: Concavity of u together with the budget constraints (8a) and (8b) imply that the left side in (9a) is a continuous, monotone and decreasing function of $y^{\mathscr{R}}$ and continuous, monotone and increasing in P_2. Hence, there is at most one $(y^{\mathscr{R}}, P_2)$ satisfying (13), which thus defines a bijective function $\phi_1 : [\phi_1^{-1}(1), \min\{1, \lambda R\}] \times [1, \min\{R, \phi_1(1)\}]$ with $d\phi_1(P_1)/dP_1 > 0$ such that for $P_2 = \phi_1(P_1)$ we have $q_1^D = q_2^D$ and $y^{\mathscr{R}} = 0$.

For any P_1, Equation (9a) defines P_2 as a monotone and increasing function of $y^{\mathscr{R}}$. Then, a sufficient condition that there is at most one $(y^{\mathscr{R}}, P_2)$ satisfying (13) and $y^{\mathscr{R}} > 0$ is that the left side in (9b) is strictly monotone in $y^{\mathscr{R}}$ while taking into account the relation between $y^{\mathscr{R}}$ and P_2 according to (9a). Let

$$\Phi_1 := \left(\frac{k_{12}}{k_{22}} \frac{1}{P_1} + \left(\frac{y^{\mathscr{R}}}{1 - y^{\mathscr{R}}} + P_1 \right) \frac{1}{1 - P_1} \frac{P_2}{P_1} + \frac{k_{11}}{k_{22}} \frac{P_2 - 1}{P_1} \right) \frac{P_2 - P_1}{P_2 - 1},$$

$$\Phi_2 := \left(\frac{k_{12}}{k_{22}} + \left(\frac{y^{\mathscr{R}}}{1 - y^{\mathscr{R}}} + P_1 \right) \frac{1}{1 - P_1} \right) \frac{P_2 - P_1}{P_2 - 1}.$$

This monotonicity holds if for all P_1 either $\Phi_1 > k_{12} - k_{11}$ or $\Phi_1 < k_{12} - k_{11}$. The sign of dP_2/dP_1 is positive if and only if $\Phi_1 > k_{12} - k_{11} > \Phi_2$. Hence, with non-increasing risk aversion, i.e. $k_{11} \geq k_{12}$, Equation (13) defines a bijective function $\phi_2 : [\max\{\varepsilon, \phi_2^{-1}(R)\}, \min\{\phi_1^{-1}(1), \phi_2^{-1}(1)\}] \times [1, R]$ with $d\phi_2(P_1)/dP_1 < 0$ such that for $P_2 = \phi_2(P_1)$ we have $q_1^D = q_2^D$ and $y^{\mathscr{R}} > 0$. □

The assumption that relative risk aversion is non-increasing eliminates possibly multiple solutions to Equation (13) and renders the monotonicity of the relation between asset prices in the two states well-defined. Non-increasing risk aversion is not necessary for many of our results though. For example, the existence of an equilibrium is as independent from this assumption as is the feasibility of state-independent liquidity demand according to (13). Yet, it has a straightforward intuition which makes it a reasonable case to look at. The possibility of a bank run not only adversely affects the expected present value of total consumption. It also creates additional volatility in consumption for both, patient and impatient consumers. For decreasing relative risk aversion, this risk is more harmful to impatient consumers. To see this, consider the solutions $(\mathbf{y}^{\mathscr{R}}, \mathbf{P}_2)$ to (13) for $y^{\mathscr{R}} > 0$. Combining (9a) and (9b) implies that these solutions also satisfy

$$\frac{u'\left(y^{\mathscr{R}} + P_1(1-y)\right)}{u'\left(\frac{y^{\mathscr{R}}+P_1(1-y)}{\lambda}\right)} - \frac{1-p}{p}\frac{P_2-1}{1-P_1} = 0. \tag{15}$$

For a higher sunspot probability p, it is optimal for consumers that the bank holds more reserves $y^{\mathscr{R}}$ if and only if an increase in $y^{\mathscr{R}}$ lowers the marginal rate of substitution between consumption in the bank run state and consumption of impatient consumers in the state without bank run. For given prices, a consumer then asks for more consumption in the bank run state and is willing to forfeit some of the liquidity insurance the bank offers when there is no bank run. This is equivalent to saying that risk aversion is decreasing $k_{11} > k_{12}$, which is a necessary and sufficient condition for the marginal rate of substitution to be decreasing in $y^{\mathscr{R}}$.

Note that non-increasing relative risk aversion is a common assumption made in models of bank runs (e.g. Fecht, 2004) or in macro models with banks (e.g. Gertler and Kiyotaki, 2015) where risk aversion is often even constant.

3.3 Zero liquidity supply

In any equilibrium without risky banks there is no liquidity demand. Hence, $q^S = 0$ must hold for $\rho = 0$. A necessary and sufficient condition thus is $y^{\mathscr{S}} = \lambda P_1/(\lambda P_1 + 1 - \lambda)$, implying $d^{\mathscr{S}} = P_1/(\lambda P_1 + 1 - \lambda)$. A safe bank will set these if and only if prices are such that $y^{\mathscr{S}} = \lambda P_1/(\lambda P_1 + 1 - \lambda)$ and $d^{\mathscr{S}} = P_1/(\lambda P_1 + 1 - \lambda)$ are a solution to its first-order condition (6), i.e. if

$$\begin{aligned}(1-P_1)\frac{u'(P_1/(\lambda P_1+1-\lambda))}{u'(R/(\lambda P_1+1-\lambda))}& \\ -R\left(\frac{1-\lambda}{\lambda}\left((1-p)\left(1-\frac{1}{P_2}\right)-p\left(\frac{1}{P_1}-1\right)\right)+(1-p)\left(1-\frac{P_1}{P_2}\right)\right)&=0.\end{aligned} \tag{16}$$

Equation (16) then defines a differentiable function h such that for all $P_2 = h(P_1)$ liquidity supply is zero. The properties of this function include $h'(P_1) < 0$, $h(1) = 1$, and $h^{-1}(R) \in]0,1[$. Moreover, $q^S > 0$ for all $P_2 < h(P_1)$ and $q^S < 0$ for all $P_2 > h(P_1)$. To understand this note that the budget constraints (5a) and (5b) imply for $y^{\mathscr{S}} = \lambda P_1/(\lambda P_1 + 1 - \lambda)$ that $dx^{\mathscr{S}}_{1,s}/dP_2 = dx^{\mathscr{S}}_{2,s}/dP_2 = 0$ for both s. Note also that the first-order condition (6) implicitly defines $y^{\mathscr{S}}$ as a function of P_2 for any given P_1. Evaluated at $y^{\mathscr{S}} = \lambda P_1/(\lambda P_1 + 1 - \lambda)$, this function satisfies $dy^{\mathscr{S}}/dP_2 < 0$. Since for every P_1 there is a unique P_2 such that $q^S = 0$, we conclude for all $P_2 < h(P_1)$ that $y^{\mathscr{S}} > \lambda P_1/(\lambda P_1 + 1 - \lambda)$ and thus $q^S > 0$ (and vice versa).

3.4 Choice between risky and safe banks

Our third step is to identify which bank is the best choice for consumers. In equilibria in which only safe banks operate, the expected utility they offer to consumers has to be weakly greater than the one offered by risky banks. Similarly, in equilibria with only risky banks these banks have to offer weakly better services to consumers than safe banks. In equilibria with both, safe and risky banks offering their services, all banks have to offer the same expected utility to consumers.

According to the Envelope theorem, indirect utilities $V^{\mathscr{R}}(P)$ and $V^{\mathscr{S}}(P)$ are characterized by

$$\frac{dV^{\mathscr{R}}(P)}{dP_2} = \quad (1-p)\,u'\left(x^{\mathscr{R}}_{2,2}\right)\frac{R}{P_2}\frac{q_2^D}{P_2} \in \begin{cases} \mathbb{R}_{++} & \text{if } \; q_2^D > 0, \\ \{0\} & \text{if } \; q_2^D = 0, \end{cases} \tag{17a}$$

$$\frac{dV^{\mathscr{S}}(P)}{dP_2} = -\,(1-p)\,u'\left(x^{\mathscr{S}}_{2,2}\right)\frac{R}{P_2}\frac{q^S}{P_2} \in \begin{cases} \mathbb{R}_{-} & \text{if } \; q^S > 0, \\ \{0\} & \text{if } \; q^S = 0, \\ \mathbb{R}_{++} & \text{if } \; q^S < 0, \end{cases} \tag{17b}$$

$$\frac{dV^{\mathscr{R}}(P)}{dP_1} = p(1-y^{\mathscr{R}})u'\left(y^{\mathscr{R}} + P_1(1-y^{\mathscr{R}})\right) > 0. \tag{17c}$$

The sign of $dV^{\mathscr{S}}(P)/dP_1$ is not clear.

Let g be a correspondence such that for $P_1 \in [\varepsilon, 1]$

$$g(P_1) = \left\{P_2 \in [1,R] \,\middle|\, V^{\mathscr{R}}(P) - V^{\mathscr{S}}(P) = 0\right\}. \tag{18}$$

Provided $q^s \geq 0$ and $g(P_1) \neq \emptyset$, the above characteristics of the indirect utilities thus imply that the correspondence g is an injective function and a consumer strictly prefers a risky bank over a safe bank if and only if $P_2 > g(P_1)$.

4 Properties of equilibria

In this section we look into the role of the sunspot probability and other key parameters for the type of equilibria that can emerge. We start with some specific examples to illustrate the nature of the different types of equilibria.

4.1 Examples

In the following examples consumers have utility $u(x) = -x^{-1}$.

Example 1 For $R = 2.5$, $\lambda = 0.4$ and $p = 0.15$ there are exactly two types of equilibria.

1. safe and risky banks

$$\begin{array}{lcrlcr} P_1 & = & 0.005137 & P_2 & = & 2.097275 \\ q_1^d & = & 0.002041 & q_2^d & = & 0.002041 \\ q^s & = & 0.506995 & \rho & = & 0.995991 \\ V^{\mathscr{R}} & = & -\ 0.781776 & V^{\mathscr{S}} & = & -\ 0.781776 \end{array}$$

2. safe banks only

$$\begin{array}{lcllcl} P_1 & \geq & 0.915570 & P_2 & = & \frac{0.6+0.4P_1}{1-\frac{1}{0.85}\frac{1-P_1}{P_1}\left(\frac{1}{P_1}+0.09\right)} \\ & & & & \leq & 1.108388 \\ q^s & = & 0 & \rho & = & 0 \\ V^{\mathscr{R}} & \leq & -\ 0.654027 & V^{\mathscr{S}} & \geq & -\ 0.654027 \end{array}$$

End of example

This example provides several interesting insights. First, equilibria can exist in which safe banks operate alongside risky banks and asset prices are volatile. Second, equilibria can exist in which only safe banks operate while asset prices are volatile. The third insight is that an economy can have multiple equilibria which can be ranked according to welfare. In the present case, the second equilibrium with only safe banks makes consumers strictly better off than the first equilibrium in which risky and safe banks coexist. Finally, there can be indeterminacy in that equilibrium asset prices and hence consumption bundles are not well defined.

Example 2 For $R = 5$, $\lambda = 0.7$ and $p = 0.17$ there is only one equilibrium with safe and risky banks.

$$\begin{array}{lcrlcr} P_1 & = & 0.306249 & P_2 & = & 1.289987 \\ q^s & = & 0.300000 & \rho & = & 0.836239 \\ V^{\mathscr{R}} & = & -\ 0.767365 & V^{\mathscr{S}} & = & -\ 0.767365 \end{array}$$

End of example

This second example shows that there may not always be multiple equilibria and that the only equilibrium can be one in which safe banks operate along with risky banks and where asset prices are determinate.

Example 3 For $R = 5$, $\lambda = 0.4$ and $p = 0.02$ there is exactly one equilibrium.

$$\begin{array}{lcrlcrl} P_1 & = & & \varepsilon & P_2 & = & 1.127551 \\ q_1^d & = & & 0 & q_2^d & = & 0 \\ \rho & = & & 1 & & & \\ V^{\mathscr{R}} & = & - & 0.470743 & V^{\mathscr{S}} & = & -\ 0.532600 \end{array}$$

End of example

In this third example the equilibrium is unique. Only risky banks exist and, when a bank run occurs, the asset price drops to or below the liquidation value.

Example 4 For $R = 5$, $\lambda = 0.4$ and $p = 0.14$ there are exactly two types of equilibria.

1. risky banks only

$$\begin{array}{lcrlcrl} P_1 & = & & \varepsilon & P_2 & = & 2.017442 \\ q_1^d & = & & 0 & q_2^d & = & 0 \\ \rho & = & & 1 & & & \\ V^{\mathscr{R}} & = & - & 0.601734 & V^{\mathscr{S}} & = & -\ 0.606790 \end{array}$$

2. safe banks only

$$\begin{array}{lcrlcrl} P_1 & \geq & & 0.992116 & P_2 & = & \frac{0.6+0.4P_1}{1-\frac{1}{0.14}\frac{1-P_1}{P_1}\left(\frac{2}{P_1}+0.084\right)} \\ & & & & & \leq & 1.0163573 \\ q^s & = & & 0 & \rho & = & 0 \\ V^{\mathscr{R}} & \leq & - & 0.521529 & V^{\mathscr{S}} & \geq & -\ 0.521529 \end{array}$$

End of example

This final example offers two additional insights. First, there are possibly equilibria in which only risky banks operate. Second, while the equilibrium with safe banks is indeterminate, the co-existing equilibrium with risky banks is not.

4.2 Financial instability

The financial system serves to facilitate productive investments and to provide liquidity insurance for consumers. It does so through banks and their interaction on asset markets. The structure of the financial system is endogenous. One possible equilibrium structure is when all banks occasionally fail simultaneously and at the same time the asset market crashes (Examples 3 and 4). Another one is when no banks fail and asset prices are the same across states (Examples 1 and 3).

Definition 2 *Suppose* $\left((y^{\mathscr{S}}, d^{\mathscr{S}}, x^{\mathscr{S}}), (y^{\mathscr{R}}, d^{\mathscr{R}}, x^{\mathscr{R}}), (P, \rho)\right)$ *is an equilibrium. Then,* (P, ρ) *is a* ***financial system****. It is an* ***unstable financial system*** *if* $P_1 = \varepsilon$*,* $P_2 > 1$ *and* $\rho = 1$ *and a* ***stable financial system*** *if* $P_1 = P_2 = 1$ *and* $\rho = 0$*.*

To understand unstable financial systems and the circumstances in which they may exist, we start with the following observation.

Lemma 2 *Suppose* $\left((y^{\mathscr{S}}, d^{\mathscr{S}}, x^{\mathscr{S}}), (y^{\mathscr{R}}, d^{\mathscr{R}}, x^{\mathscr{R}}), (P, \rho)\right)$ *is an equilibrium and let*

$$\hat{p} := \frac{R-1}{R-1+u'\left(\frac{\lambda R}{\lambda R+1-\lambda}\right)/u'\left(\frac{R}{\lambda R+1-\lambda}\right)}.$$

Then the financial system cannot be unstable if $p > \hat{p}$*.*

Proof: $\rho = 1$ implies $Q^S = 0$. Accordingly, for equilibria with $\rho = 1$ it requires $\lambda d^{\mathscr{R}} - y^{\mathscr{R}} = 0$ and either $1 - y^{\mathscr{R}} = 0$ or $P_1 \leq \varepsilon$. We can rule out $1 - y^{\mathscr{R}} = 0$ because state-independence of liquidity demand requires $y^{\mathscr{R}}$ to solve

$$\frac{u'\left(\frac{y^{\mathscr{R}}+P_1(1-y^{\mathscr{R}})}{\lambda}\right)}{u'\left(\frac{R}{P_2}\frac{P_2-P_1}{1-\lambda}(1-y^{\mathscr{R}})\right)} - \frac{R}{P_2} = 0,$$

and concavity of u implies an upper bound on $y^{\mathscr{R}}$ given by $y^{\mathscr{R}} \leq \lambda R/(\lambda R + 1 - \lambda) < 1$. Hence, an equilibrium exists only if $P_1 \leq \varepsilon$ and $f(\varepsilon) \neq \emptyset$, i.e. there is some $(y^{\mathscr{R}}, P_2) \in [0, \lambda R/(\lambda R + 1 - \lambda)] \times [1, R]$ satisfying

$$\begin{aligned} \frac{u'\left(y^{\mathscr{R}}/\lambda\right)}{u'\left(\frac{R(1-y^{\mathscr{R}})}{1-\lambda}\right)} &= \frac{R}{P_2}, \\ \frac{u'\left(y^{\mathscr{R}}\right)}{u'\left(\frac{R(1-y^{\mathscr{R}})}{1-\lambda}\right)} &= \frac{R}{P_2}\frac{1-p}{p}(P_2-1). \end{aligned}$$

Let Y_1 be the solution to the first equation for a given P_2. Then, $\lim_{P_2\to 1} Y_1 = y^*$, $\lim_{P_2\to R} Y_1 = \lambda R/(\lambda R + (1-\lambda))$ and $dY_1/dP_2 > 0$. Let Y_2 be the solution to the second equation for a given P_2. Then, $\lim_{P_2\to 1} Y_2 = 1$, $\lim_{P_2\to R} Y_2 = \tilde{y} \in (0,1)$ and $dY_2/dP_2 < 0$ where $\tilde{y}$ is implicitly defined by

$$\frac{u'(\tilde{y})}{u'\left(\frac{R(1-\tilde{y})}{1-\lambda}\right)} = \frac{1-p}{p}(R-1).$$

Since $y^* < 1$, there is no $f(\varepsilon) \in [0, \lambda R/(\lambda R + 1 - \lambda)] \times [1, R]$ if

$$\frac{u'\left(\frac{\lambda R}{\lambda R+(1-\lambda)}\right)}{u'\left(\frac{R}{\lambda R+(1-\lambda)}\right)} > \frac{1-p}{p}\,(R-1)\,,$$

or, equivalently, if $p > \hat{p}$. □

A necessary condition for unstable financial systems to exist is that liquidity demand is zero in both states. There is no liquidity demand in the sunspot state if and only if the asset price is not larger than the physical liquidation value of assets. Liquidity demand in the no-sunspot state is zero only if the total payout to impatient consumers is just equal to the reserve holdings of the risky bank. For high levels of reserves, and thus high payouts to impatient consumers, the budget constraint implies that the payout to patient consumers will be low. The marginal rate of substitution between consumption when patient and when impatient would be lower than the rate of return on the long asset as of date 1 unless the price of the long asset is high. This is because a high asset price means a high consumption for patient consumers and a low rate of return on holding the long asset between date 1 and date 2. Since the price of the long asset is bounded above by R, the return on the long asset has a lower bound and consumption when patient has an upper bound. Hence, there is an upper bound on the reserve holdings above which it is better for consumers to have some payout in excess of the bank's reserves when becoming impatient. Liquidity demand would then no longer be zero in the no-sunspot state. Since optimal reserves are the larger the higher is the sunspot probability, they are sufficiently small to allow for state-independent liquidity demand if and only if the sunspot probability is below some threshold $\hat{p}$.

The upper bound $\hat{p}$ on the sunspot probability is strictly smaller than $(R-1)/R < 1$ and depends on the characteristics of the economy. It is the lower the smaller the share of early consumers is. Provided liquidity demand is zero, fewer impatient consumers implies that the maximum payoff to consumers in the state without a bank run is larger while the maximum payoff in case of a bank run is smaller. Consumers will find this consumption profile efficient only if they are less likely to experience a bank run. The effects of the return on the long asset R on $\hat{p}$ are generally not clear-cut, for there are two effects possibly working in opposite directions. On the one hand, a larger R eases the upper bound on P_2. This allows patient consumers to get more for any given reserve holdings, and in order to rebalance their optimal consumption profile consumers want to consume more when impatient too. Banks can offer this even without resorting to the asset market in the no-sunspot state by holding more reserves. Hence, the sunspot probability which determines optimal reserve holdings can be higher. On the other hand, a larger R also changes the optimum consumption profile for consumers in case of a run compared to what they get as late consumers in case there is no run. If risk aversion is non-increasing, however, increasing reserves according to the first effect is more than enough to re-balance the marginal utilities across those states. The threshold for the sunspot probability then clearly increases. Note, if u exhibits constant relative risk aversion with $k(x) = K$, we have $\hat{p} = (R-1)/\left(R-1+\lambda^{-K}\right)$ with $d\hat{p}/dR > 0$ and $d\hat{p}/dK < 0$.

Theorem 2 *Assume utility is bounded above. There exists a $\bar{p} > 0$ such for all $p < \bar{p}$ an unstable financial system is an equilibrium. Such equilibrium is locally isolated. Welfare approaches the full-information first-best if the sunspot probability converges to zero.*

Proof: A necessary condition for $V^{\mathscr{R}}(P) \geq V^{\mathscr{S}}(P)$ provided $P_1 \leq \varepsilon$ and $P_2 = \phi_2^{-1}(\varepsilon)$ is $\lim_{x\to\infty} u(x)^{-1} > 0$, else safe banks always offer higher expected utility than risky banks at these prices. Note that

$$\frac{dV^{\mathscr{R}}(P)}{dp} = \frac{dV^{\mathscr{R}}(P)}{dP_2}\frac{dP_2}{dp} + \frac{dV^{\mathscr{R}}(P)}{dy^{\mathscr{R}}}\frac{dy^{\mathscr{R}}}{dp} + u(y^{\mathscr{R}}) - \left(\lambda u\left(\tfrac{y^{\mathscr{R}}}{\lambda}\right) + (1-\lambda)u\left(\tfrac{R(1-y^{\mathscr{R}})}{1-\lambda}\right)\right) < 0,$$

because $dV^{\mathscr{R}}(P)/dP_2 = 0$ for $q_2^D = 0$ and $dV^{\mathscr{R}}(P)/dy = 0$ for $y = y^{\mathscr{R}}$, and because the properties of u imply $u(y^{\mathscr{R}}) \leq u(\lambda R/(\lambda R + 1 - \lambda))$ and $\lambda u(y^{\mathscr{R}}\lambda^{-1}) + (1-\lambda)u(R(1-y^{\mathscr{R}})/(1-\lambda)) \geq u(R(\lambda R + 1 - \lambda))$ since $y^* \leq y^{\mathscr{R}} \leq \lambda R/(\lambda R + 1 - \lambda)$. Moreover, $\lim_{p\to 0} V^{\mathscr{R}}(P) = \lambda u(y^*/\lambda) + (1-\lambda)u(R(1-y^*)/(1-\lambda))$. While the effects of p on $V^{\mathscr{S}}(P)$ are not clear, $V^{\mathscr{S}}(P) < \lambda u(y^*/\lambda) + (1-\lambda)u(R(1-y^*)/(1-\lambda))$ for all p. Therefore, by the intermediate value theorem there is a $\bar{p}$ such that $V^{\mathscr{R}}(P) > V^{\mathscr{S}}(P)$ for $p < \bar{p}$ provided $\lim_{x\to\infty} u(x)^{-1} > 0$. The equilibrium is locally isolated because for $p \leq \bar{p}$ there is a unique solution to (13) given by

$$\begin{aligned} pu'\left(y^{\mathscr{R}}\right) - (1-p)\left(u'\left(y^{\mathscr{R}}/\lambda\right) - Ru'\left(\frac{R\left(1-y^{\mathscr{R}}\right)}{1-\lambda}\right)\right) &= 0, \\ \frac{Ru'\left(\frac{R\left(1-y^{\mathscr{R}}\right)}{1-\lambda}\right)}{u'\left(y^{\mathscr{R}}/\lambda\right)} &= P_2. \end{aligned}$$

□

Unstable financial systems can exist provided coordination failures among consumers are possible. Without coordination failures among consumers, this cannot happen in limit economies where fundamental risks converge to zero (Allen and Gale, 2004a). All banks being risky in equilibrium, however, necessarily goes along with a concurrence of bank runs and a total devaluation of assets (up to physical liquidation). The reason is that all assets will be put up for sale simultaneously while there is no liquidity in the market. This can only be an equilibrium if the prospect of an (almost) unlimited return cannot induce some banks to hold sufficient reserves to fend off a bank run caused by coordination failures and to buy assets from other banks at fire sale prices.

For this to happen, one necessary condition is that utility is bounded above. Otherwise the prospect of (almost) unlimited returns, even with only slim chances, would make consumers with a safe bank strictly better off than with a risky bank. Utility being bounded above holds for many risk-preferences with relative risk aversion greater than one, particularly for those with non-increasing relative risk aversion. The other necessary condition is that the sunspot probability is sufficiently close to zero. Then liquidity demand of risky banks is zero in both states and the prospect of (almost) unlimited returns in the sunspot state does not sufficiently compensate consumers for the efficiency loss associated with the comparatively large reserves a safe bank would have to hold in order to avoid a bank run.

When a financial system is unstable, the real outcome is well defined because the equilibrium is locally isolated. If the sunspot probability approaches zero, the expected utility converges to the full-information first-best as $y^{\mathscr{R}}$ approaches y^*. The equilibrium price P_1 is always (weakly) below ε and with the sunspot probability approaching zero, the equilibrium price P_2 converges to one. In that sense, an equilibrium without fundamental risk and in which banks only fail for fundamental reasons (as the one considered in Allen and Gale, 2004a) forms the limit equilibrium in a world without fundamental risk but with bank runs caused by coordination failures. However, the possibility of coordination failures among depositors also eliminates the indeterminacy regarding the asset prices that exists if such failures can be rule out.

4.3 Financial stability

Our next result is about the existence and properties of stable financial systems.

Theorem 3 *There exists a $\check{p} < 1$ such that for all $p > \check{p}$ a stable financial system is an equilibrium. Welfare with a stable financial system is always strictly smaller than in the full-information first-best world.*

Proof: $\rho = 0$ requires $q^S = 0$. Absence of asset price volatility requires $P_1 = P_2 = 1$. For safe banks, the budget constraints (5a) and (5b) then imply $d^{\mathscr{S}} = 1$, $x^{\mathscr{S}}_{2,1} = R$ and $x^{\mathscr{S}}_{2,2} = R$. For risky banks, $d^{\mathscr{R}}$ solves

$$u'(d^{\mathscr{R}}) = Ru'\left(R\tfrac{1-\lambda d^{\mathscr{R}}}{1-\lambda}\right),$$

implying $d^{\mathscr{R}} \in \left[1, \lambda^{-1}\right)$ and $R\frac{1-\lambda d^{\mathscr{R}}}{1-\lambda} \in (1, R]$ for $k(x) > 1$. Therefore, $x^{\mathscr{R}}_{1,1} = 1 = x^{\mathscr{S}}_{1,1}$, $x^{\mathscr{R}}_{1,2} \geq 1 = x^{\mathscr{S}}_{1,2}$, $x^{\mathscr{R}}_{2,1} = 1 < R = x^{\mathscr{S}}_{2,1}$ and $x^{\mathscr{R}}_{2,2} \leq R = x^{\mathscr{S}}_{2,1}$. Let

$$X(p) = (1-p)\,\lambda u\left(x^{\mathscr{R}}_{1,2}\right) + (1-p)\,(1-\lambda)\,u\left(x^{\mathscr{R}}_{2,2}\right) + pu(1),$$

and $\check{p}$ be a solution to

$$\lambda u(1) + (1-\lambda)\,u(R) = X(p)$$

Since $\lambda u\left(x^{\mathscr{R}}_{1,2}\right) + (1-\lambda)\,u\left(x^{\mathscr{R}}_{2,2}\right) > \lambda u(1) + (1-\lambda)\,u(R)$ for $k(x) > 1$, $u(1) < \lambda u(1) + (1-\lambda)\,u(R)$ and $X' < 0$, there is a unique $\check{p} < 1$ such that $V^{\mathscr{S}}(P) \geq V^{\mathscr{R}}(P)$ for $P = (1,1)$ if and only if $p \geq \check{p}$. Since $k > 1$, expected utility satisfies $\lambda u(1) + (1-\lambda)u(R) < \lambda u(y^*/\lambda) + (1-\lambda)u(R(1-y^*)/(1-\lambda))$. □

In a stable financial system, neither do banks go bust nor do asset prices fluctuate as a result of sunspots. Asset prices being equal across states have two major implications. First, an individual safe bank's reserves are indeterminate, as structuring its portfolio at $t = 0$ is as good as structuring it at $t = 1$. A bank can simply buy and sell the long asset at $t = 1$ as a unit of reserves is as much worth as a unit of the long asset at both dates. In aggregate, however, the banking sector makes provisions against bank runs driven by coordination failures. There are just sufficient reserves in the banking sector to allow all banks paying out all depositors at $t = 1$, i.e. $\lambda d^{\mathscr{S}} = y^{\mathscr{S}}$. Second, while there may be a trade of assets at $t = 1$, it does not affect the consumption for impatient or patient consumers. The market value of a bank's assets at $t = 1$ is always one regardless how it is structured. Hence, impatient consumers always get one unit of consumption and patient consumers always get R units.

For a risky bank, the individual reserve holdings would also be indeterminate for $P_1 = P_2 = 1$. However, it would provide a better risk sharing in case there is no bank run by offering higher consumption to impatient consumers and lower consumption to patient consumers. The cost is that, relative to what a safe bank offers, all consumers get less in case of a bank run. Hence, if the sunspot probability is sufficiently large, safe banks outperform risky banks.

It is useful to further classify the banking sector of a financial system.

Definition 3 *A **stable banking sector** is a financial system with* $\rho = 0$ *and a **mixed banking sector** is a financial system with* $\rho \in]0,1[$.

Theorem 4 *Suppose the equilibrium* $\left((y^{\mathscr{S}}, d^{\mathscr{S}}, x^{\mathscr{S}}), (y^{\mathscr{R}}, d^{\mathscr{R}}, x^{\mathscr{R}}), (P, \rho)\right)$ *is a stable financial system. If* $V^{\mathscr{S}}(P) > V^{\mathscr{R}}(P)$*, there is a stable banking sector such that*

- *asset prices and consumption are indeterminate;*
- *asset prices are strictly bounded away from the physical liquidation value.*

Proof: In any equilibrium with safe banks only, $P_2 = h(P_1)$ must hold. Continuity of h implies there exists a continuum of equilibrium prices which support equilibria with stable banking sectors provided $V^{\mathscr{S}}(1,1) > V^{\mathscr{R}}(1,1)$, i.e. if $p > \check{p}$. Different asset prices are associated with different consumption bundles according to Equation (6). Since arbitrage-free equilibrium requires $1 \le P_2 \le R$ and because $h^{-1}(R) > 0$, P_1 is strictly bounded away from ε.

□

If the sunspot probability is sufficiently large, an economy may not necessarily have a stable financial system despite no bank offering its liquidity service to consumers will ever suffer a bank run. This is because asset prices may well be volatile, although a total market crash cannot happen. Just like in Allen and Gale (2004a), asset prices are indeterminate in this equilibrium. However, the possibility of coordination failures induces safe banks to hold more reserves and the amount of reserves depends on the extent of the asset price volatility. The more volatile they are, the lower is the asset price in the sunspot state and the tighter is the constraint of safe banks. As this induces safe banks to hold more liquid reserves, asset price volatility has an impact on welfare which is the stronger the more prices fluctuate in equilibrium. From a welfare perspective, with the banking sector being stable, consumers could be made better off if policy makers would commit to stabilize asset prices when they tend to fall below their fundamentals. The anticipation of such interventions could serve as a coordination device and the economy converges to a stable financial system.

5 Real effects of financial instability

The equilibrium structure of the financial system has consequences for the efficiency of the liquidity insurance it provides for consumers. The efficiency of the liquidity insurance then influences the decisions of consumers to provide funds to the financial sector. In that sense, the structure of the financial system matters for the real economy beyond its ability to insure against idiosyncratic liquidity risks even in the absence of any fundamental risk. In this section we shed some light on these links.

5.1 Financial systems and efficiency of liquidity insurance

We compare consumer welfare in different financial systems for a given set of fundamentals. This focus on multiple equilibria allows to elicit the role of the instability of the financial system for the real economy. For the sake of clarity and simplicity we assume from now on that consumers have risk preferences with constant relative risk aversion.

If there is a mixed banking sector, asset prices have to be volatile but will be bounded away from their physical liquidation value. Hence, asset prices would not be volatile if and only if $P_1 = P_2 = 1$. However, there would be no liquidity supply for $h(1) = 1$. Market clearing would then require that liquidity demand is also zero. However, $q_1^D = q_2^D = 0$ holds if and only if $P_2 = \phi_2(\varepsilon)$. Hence, market clearing with $\rho \in]0,1[$ cannot hold for $P_2 = P_1$.

For relative risk aversion being constant, equilibria can be ranked according to the efficiency of the liquidity insurance the financial sector provides by a simple rule.

Theorem 5 *Assume relative risk aversion is constant. Comparing any two financial systems, welfare is higher in the equilibrium in which the asset price is higher in the sunspot state.*

Proof: Provided $k(x)$ is constant, f is a function for all $f(P_1) \neq \emptyset$ (see Lemma 1).
Welfare for $\rho \in]0,1]$: For $f(P_1) = P_2$, indirect utility is

$$V^{\mathscr{R}}(P_1) = pu\left(y^{\mathscr{R}} + P_1\left(1 - y^{\mathscr{R}}\right)\right) + (1-p)\,\lambda u\left(\tfrac{y^{\mathscr{R}} + P_1(1-\hat{y})}{\lambda}\right) + (1-p)\,(1-\lambda)\,u\left(\tfrac{R}{P_2}\tfrac{P_2 - P_1}{1-\lambda}\left(1 - y^{\mathscr{R}}\right)\right),$$

with $(y^{\mathscr{R}}, P_2) = f(P_1)$. According to the Envelope theorem it follows

$$\frac{dV^{\mathscr{R}}(P_1)}{dP_1} = \begin{cases} \dfrac{\left(\frac{k_{22}}{1-P_1} + \frac{k_{11}}{P_1} + \frac{P_2-1}{1-P_1}\frac{1}{P_1}\right)(1-p)\left(1-y^{\mathscr{R}}\right)(P_2-P_1)\,u'\left(x_{2,2}^{\mathscr{R}}\right)}{k_{22} + \frac{P_2-P_1}{P_1}} & \text{for} \quad y^{\mathscr{R}} = 0, \\[2ex] \dfrac{(1-p)\left(1-y^{\mathscr{R}}\right)\left(k_{12} + \left(k_{22}\left(\frac{y^{\mathscr{R}}}{1-y^{\mathscr{R}}} + P_1\right)\frac{1}{1-P_1} + k_{11}\right)\frac{P_2-1}{1-P_1}\right)u'\left(x_{2,2}^{\mathscr{R}}\right)}{(k_{11}-k_{12})\,k_{22}\frac{P_1}{P_2-P_1} + k_{12}\frac{1}{P_2-1} + k_{22}\left(\frac{y^{\mathscr{R}}}{1-y^{\mathscr{R}}} + P_1\right)\frac{1}{1-P_1}\frac{P_2}{P_2-1} + k_{11}} & \text{for} \quad y^{\mathscr{R}} > 0, \end{cases}$$

with $x_{2,2}^{\mathscr{R}} = \frac{R}{P_2}\frac{P_2-P_1}{1-\lambda}(1-y^{\mathscr{R}})$. It is strictly positive for $y^{\mathscr{R}} = 0$. For $y^{\mathscr{R}} > 0$ it is positive if and only if

$$\left(\frac{k_{12}}{k_{22}}\frac{1}{P_1} + \left(\frac{y^{\mathscr{R}}}{1-y^{\mathscr{R}}} + P_1\right)\frac{1}{1-P_1}\frac{P_2}{P_1} + \frac{k_{11}}{k_{22}}\frac{P_2-1}{P_1}\right)\frac{P_2-P_1}{P_2-1} > k_{12} - k_{11},$$

which holds for constant $k(x)$. Therefore, the welfare of any equilibrium financial system with $\rho \in]0,1]$ is the higher the higher is P_1.

Welfare for $\rho = 0$: For $h(P_1) = P_2$, indirect utility is

$$V^{\mathscr{S}}(P_1) = \lambda u\left(\frac{P_1}{\lambda P_1 + 1 - \lambda}\right) + (1-\lambda)\,u\left(\frac{R}{\lambda P_1 + 1 - \lambda}\right).$$

Applying the Envelope theorem, taking $P_2 = h(P_1)$ into account, it follows

$$\frac{dV^{\mathscr{S}}(P_1)}{dP_1} = \lambda u'\left(\frac{P_1}{\lambda P_1 + 1 - \lambda}\right)\frac{1-\lambda}{(\lambda P_1 + 1 - \lambda)^2} - (1-\lambda)\,u'\left(\frac{R}{\lambda P_1 + 1 - \lambda}\right)\frac{\lambda R}{(\lambda P_1 + 1 - \lambda)^2},$$

which is positive for all $P_1 \in [h^{-1}(R), 1]$. This is because $u'(1) \geq Ru'(R)$ (since $k(x) > 1$) and $\frac{d}{dP_1}(u'(\frac{P_1}{\lambda P_1+1-\lambda}) - Ru'(\frac{R}{\lambda P_1+1-\lambda})) < 0$ (since $u'' < 0$) together imply $u'(\frac{P_1}{\lambda P_1+1-\lambda}) \geq Ru'(\frac{R}{\lambda P_1+1-\lambda})$. Therefore, the welfare of any equilibrium financial system with $\rho = 0$ is the higher the higher is P_1.

Conclusion: In any equilibrium, $P_2 = \min\{\max\{\phi_1(P_1), \phi_2(P_2)\}, h(P_1)\}$. Since $dV^{\mathscr{R}}(P_1)/dP_1 > 0$ for $P_2 = \phi_1(P_1)$ and for $P_2 = \phi_2(P_1)$, as well as $dV^{\mathscr{S}}(P_1)/dP_1 > 0$ for $P_2 = h(P_1)$, comparing any two feasible equilibrium financial systems, welfare is higher in the equilibrium in which the asset price is higher in the sunspot state. □

If there are multiple equilibria, they differ with respect to the asset price in the no-sunspot state. The higher the asset price, the higher is the welfare the financial system offers for consumers. For example, if both, a mixed banking sector and a stable banking sector form an equilibrium, the latter is better for consumers than the former (see Example 1). There will be indeterminacy with a stable banking sector, but the minimum welfare it offers is still higher. The reason is that the possibility of a coordination failure among consumers comes along with a substantial asset price drop triggered by such a bank run. A similar result holds if an unstable financial system can emerge as well as a stable banking sector (see Example 4).

5.2 Financial systems and capital formation

In a second step we now analyze the consumers' decision about how much funds will be provided to the financial sector to invest on their behalf. Consider a stationary overlapping generations economy with production. There is labor, capital and a non-perishable consumption good. Time is discrete and extends from $-\infty$ to $+\infty$. At every even date t a continuum of identical consumers of mass one is born, who live for two subsequent dates $t+1$ and $t+2$. A consumer is described by her consumption set $X \in \mathbb{R}^3_+$ and an endowment of labor $\ell = 1$ at t which is inelastically supplied. Lifetime utility is

$$\Omega(x_0, x_1, x_2) = \omega(x_0) + U(x_1, x_2),$$

with ω being twice differentiable with $\omega' > 0$, $\omega'' < 0$ and $\lim\limits_{x \to 0} \omega' = \infty$, while $U(x_1, x_2)$ is as in Equation (1).

There is a continuum of firms of mass one. At even dates t, firms transform capital K_{t-2} and labor L_t into the consumption good at t. Capital fully depreciates either due to physical liquidation at the interim date $t-1$ or after completion of the production cycle at t. A firm is described by its constant returns to scale production function $Z : \mathbb{R}^2_+ \to \mathbb{R}_{++}$ with $Y = Z(K, L)$. Z is of Cobb-Douglas type and there is an externality in production such that each firm's production function is given by

$$Y = A\bar{K}^{1-\alpha} K^{\alpha} L^{1-\alpha},$$

with $\alpha \in (0, 1)$ being the output elasticity of capital, $\bar{K}$ the aggregate capital stock and A the total factor productivity. Aggregate production is then given by a standard AK function

$$Y = A\bar{K}.$$

Perfect competition among firms ensures that capital and labor are paid according to their marginal product, i.e. $W = (1-\alpha)A\bar{K}$ and $R = \alpha A$. For capital to be more productive than holding reserves it has to be $A > \alpha^{-1}$.

At t, in the first stage of her life, a consumer works and receives a wage W, consumes x_0 and saves the remainder $s = W - x_0$. Savings for the second stage of her life, which starts at $t+1$, can be either storing consumption goods or holding deposits in banks. Deposit contracts and the bank's portfolio choice are as described in section 2.2. Banks can invest in capital K which they rent to firms for the rental price R. Capital is thus the long asset. Banks can also hold reserves by storing what consumers have deposited, which is the short asset.

For the sake of simplicity we assume $\omega(x) = u(x) = -x^{-1}$ for all x. Because relative risk aversion is constant the portfolio structure of banks and hence the equilibrium asset prices are independent of the amount consumers put into the bank. A consumer's welfare then is

$$\Omega(x_0, x_1, x_2) = -\frac{1}{x_0} + \frac{V(\mathbf{P})}{s},$$

with $V(\mathbf{P})$ being the indirect utility consumers get when saving one unit using a financial system as described in section 4. Optimal consumption when young $\bar{x}_0$ and savings $\bar{s}$ are thus given by $\bar{s} = W\sqrt{-V(\mathbf{P})}/(1+\sqrt{-V(\mathbf{P})})$ and $\bar{x}_0 = W/(1+\sqrt{-V(\mathbf{P})})$.

Steady states are equilibria where capital and production grow at a constant rate. If existent, they satisfy

$$\frac{K_t}{K_{t-2}} = (1-\bar{y})(1-\alpha)A\frac{\sqrt{-V(\mathbf{P})}}{1+\sqrt{-V(\mathbf{P})}}, \tag{19}$$

with $\bar{y} = \rho y^{\mathscr{R}} + (1-\rho)y^{\mathscr{S}}$ being the aggregate liquidity holdings of the banking sector. Equation (19) is central for the real effects of financial instability for there are two effects, potentially working in opposite directions. Consider the case with two equilibria: a stable financial system and a mixed banking sector (as in Example 1). Zero-liquidity supply in the stable financial system requires $y^{\mathscr{S}} = \lambda$ which is below the full-information first-best. With a mixed banking sector, aggregate liquidity holdings can be higher or lower than the full-information first-best. If the asset price in the sunspot state is close to the physical liquidation value, aggregate reserves are more likely to be above the full-information first-best (see Lemma 2). Then, of any unit of savings deposited with banks, a larger share will be used to form capital in the stable financial system. However, the indirect utility with a stable financial system will also be higher which implies lower savings (see Lemma 5). Hence, the overall effect of a more stable financial system on capital growth is ambivalent.

Things are different though for a stable banking sector which is associated with indeterminacy. There, liquidity holdings of safe banks as well as the indirect utility consumers get from investing one unit in the banking sector are the higher the lower the volatility of asset prices is. Therefore, the growth rate of capital in a stable financial system is unambiguously lower than with a stable banking sector but volatile asset prices.

Steady states cannot exist if the unstable financial system is the equilibrium in the second stage. Suppose the sunspot probability is very small and the indirect utility consumers get in the second stage is close to the full-information first-best. When bank runs are triggered as a result of coordination failures among consumers, all capital will be physically liquidated. This liquidation then creates an externality for future generations who have no capital to work with. Although not explicitly considered here, one could think of consumers to start working with a rather unproductive technology that requires only labor. This would then kick-start the economy after a the financial crash and hence capital starts growing again.

Note that only unstable financial systems can create such externalities for the following generations. The reason is that physical liquidation of capital never occurs if there are at least some safe banks in equilibrium. The capital stock then changes hands rather than being liquidated and the asset price clears the market. Capital will thus be available for the next generation to work with. This adds to the findings by Ennis and Keister (2003). There, the only option for banks is to physically liquidate in a bank run as there is no asset market. Accordingly, every bank run makes bank-financed capital obsolete. When asset markets exist, this effect is at work only if all banks are risky.

6 Concluding remarks

Simultaneous asset market crashes and bank failures may not only occur in response to a change in fundamentals. In a banking model with asset markets we have shown that coordination failures among depositors triggered by sunspots can have similar effects. In equilibrium, risky banks which expose themselves to such bank runs may well exist even in the absence of fundamental risks. There are other types of equilibria in which safe banks exist. These banks hold portfolios that take away the incentives for consumers to coordinate on bank runs. Consumption by at least some patient and impatient consumers is stochastic if risky banks exist and provides too little liquidity insurance when safe banks exist. In any case, consumption deviates from the full-information first-best.

The instability of financial systems as they emerge in equilibrium has consequences for the real economy. These are driven by two effects which possibly work in opposite directions. One is that in more stable systems savings are more likely to be invested in productive long-term capital formation. However, the amount consumers save may vary depending on financial system.

For policy makers, the possibility of coordination failures makes it difficult to decide which financial system should be considered better. Suppose they focus only on the welfare that consumers get from the banking sector for given savings. Then among all equilibrium financial systems, those with a lower drop in asset prices (possibly coinciding with bank runs) tend to be better, at least for non-increasing relative risk aversion. However, this is not to say that unstable financial are per se bad. If coordination failures are highly unlikely, the only equilibrium that exists is one in which occasionally all banks go bust and prices of assets fall to or below their physical liquidation value. But these unstable financial systems almost implement the full-information first-best. The drawback is that they may create an externality on future generations of consumers.

We have considered a rather limited set of options for consumers to interact with banks. A key feature in the world financial crisis has been that funds withdrawn from one bank were re-deposited in another bank. This migration of deposits when banks get into distress is a channel through which the available aggregate liquidity is distributed in times of systemic crises. As this channel would work parallel, and possibly interacts with asset markets, the implications of deposit migration on asset prices and the risk-taking behavior of banks in equilibrium remains to be explored.

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A Supersafe banks

This appendix shows that there can be no supersafe banks if $-\frac{u''(x)}{u'(x)}x > 1$. Suppose (3) would never be binding such that the associated FOC are

$$\begin{aligned} u'(d) &= R\left(u'\left(\tfrac{R}{P_1}\tfrac{y+P_1(1-y)-\lambda d}{(1-\lambda)}\right)\tfrac{p}{P_1} + u'\left(\tfrac{R}{P_2}\tfrac{y+P_2(1-y)-\lambda d}{(1-\lambda)}\right)\tfrac{1-p}{P_2}\right), \\ u'(c_{2,1}) &= -\tfrac{1-p}{p}\tfrac{P_1}{1-P_1}\tfrac{1-P_2}{P_2}u'(c_{2,2}). \end{aligned}$$

There is a d which maximizes expected utility and satisfies $d < y + P_1(1-y)$ if

$$u'(y+P_1(1-y)) < p\tfrac{R}{P_1}u'\left(\tfrac{R}{P_1}(y+P_1(1-y))\right) + (1-p)\tfrac{R}{P_2}u'\left(\tfrac{R}{P_2}\tfrac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}\right).$$

To show that this cannot be, we argue that

$$\tfrac{R}{P_1}u'\left(\tfrac{R}{P_1}(y+P_1(1-y))\right) > u'(y+P_1(1-y)), \tag{20}$$

and

$$\tfrac{R}{P_2}u'\left(\tfrac{R}{P_2}\tfrac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}\right) > u'(y+P_1(1-y)), \tag{21}$$

cannot be true. Condition (20) cannot hold for $-\frac{u''(x)}{u'(x)}x > 1$ since

$$\tfrac{R}{P_1}u'\left(\tfrac{R}{P_1}(y+P_1(1-y))\right) = u'(y+P_1(1-y)) + \tfrac{1}{y+P_1(1-y)}\int_{y+P_1(1-y)}^{\frac{R}{P_1}(y+P_1(1-y))}\left[u'(x)+xu''(x)\right]dx.$$

As regards condition (21), consider first the differential equation

$$\begin{aligned} u'(y+P_1(1-y)) = {} & \tfrac{R}{P_2}\tfrac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)(y+P_1(1-y))}u'\left(\tfrac{R}{P_2}\tfrac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}\right) \\ & -\tfrac{1}{y+P_1(1-y)}\int_{y+P_1(1-y)}^{\frac{R}{P_2}\frac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}}\left[u'(x)+xu''(x)\right]dx. \end{aligned}$$

Condition (21) would hold if

$$\begin{aligned} u'\left(\tfrac{R}{P_2}\tfrac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}\right)\tfrac{R}{P_2} > {} & \tfrac{R}{P_2}\tfrac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)(y+P_1(1-y))}u'\left(\tfrac{R}{P_2}\tfrac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}\right) \\ & -\tfrac{1}{y+P_1(1-y)}\int_{y+P_1(1-y)}^{\frac{R}{P_2}\frac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}}\left[u'(x)+xu''(x)\right]dx. \end{aligned}$$

Rearranging terms gives

$$\tfrac{R}{P_2}u'\left(\tfrac{R}{P_2}\tfrac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}\right)\left(\tfrac{(P_2-P_1)(1-y)}{(1-\lambda)}\right) < \int_{y+P_1(1-y)}^{\frac{R}{P_2}\frac{(1-\lambda)y+(P_2-\lambda P_1)(1-y)}{(1-\lambda)}}\left[u'(x)+xu''(x)\right]dx.$$

However, this cannot be if $-\frac{u''(x)}{u'(x)}x > 1$ because $P_1 \le P_2$.